

# Planar Harmonic and Monogenic Polynomials of Type A

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## Abstract

Harmonic polynomials of type A are polynomials annihilated by the Dunkl Laplacian associated to the symmetric group acting as a reflection group on  $\mathbb{R}^N$ . The Dunkl operators are denoted by  $T_j$  for  $1 \leq j \leq N$ , and the Laplacian  $\Delta_\kappa = \sum_{j=1}^N T_j^2$ . This paper finds the homogeneous harmonic polynomials annihilated by all  $T_j$  for  $j > 2$ . The structure constants with respect to the Gaussian and sphere inner products are computed. These harmonic polynomials are used to produce monogenic polynomials, those annihilated by a Dirac-type operator.

## 1 Introduction

The symmetric group  $\mathcal{S}_N$  acts on  $x \in \mathbb{R}^N$  as a reflection group by permutation of coordinates. The group is generated by reflections in the mirrors  $\{x : x_i = x_j, i < j\}$ . The function  $w_\kappa(x) = \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\kappa}$  with parameter  $\kappa$  is invariant under this action and for  $\kappa > -\frac{1}{N}$  there are several measures that incorporate  $w_\kappa$  and give rise to interesting orthogonality structures. The corresponding measure on the  $N$ -torus is related to the Calogero-Sutherland quantum-mechanical model of  $N$  identical particles on the circle with  $1/r^2$  interaction potential, and the measure  $w_\kappa(x) e^{-|x|^2/2} dx$  is related to the model of  $N$  identical particles on the line with  $1/r^2$  interactions and harmonic confinement. This paper mainly concerns the measure on the unit sphere in  $\mathbb{R}^N$  for which there is an orthogonal decomposition

involving harmonic polynomials. In the present setting *harmonic* refers to the Laplacian operator  $\Delta_\kappa$  produced by the type- $A$  Dunkl operators.

For  $x \in \mathbb{R}^N$  and  $\{i, j\} \subset \{1, 2, \dots, N\}$  set  $x(i, j) = (\dots, \overset{i}{x_j}, \dots, \overset{j}{x_i}, \dots)$ , that is, entries  $\#i$  and  $\#j$  are interchanged.

**Definition 1** For a polynomial  $f$  and  $1 \leq i \leq N$

$$T_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1, j \neq i}^N \frac{f(x) - f(x(i, j))}{x_i - x_j},$$

$$\Delta_\kappa f(x) := \sum_{i=1}^N T_i^2.$$

The (*Dunkl*) operators  $T_i$  mutually commute and map polynomials to polynomials. The background for the theory can be found in the treatise [4, Ch. 6, Ch. 10.2]. An orthogonal basis for  $L^2(\mathbb{R}^N, w_\kappa(x) e^{-|x|^2/2} dx)$  can be defined in terms of products  $f(x) L_n^\lambda(|x|^2/2)$  where  $f$  comes from an orthogonal set of harmonic homogeneous polynomials and the Laguerre polynomial index  $\lambda = \deg f - 1 + \frac{N}{2}((N-1)\kappa + 1)$ . However attempts to explicitly construct harmonic polynomials run into technical complications, presumably due to the fact that the sign-changes (example:  $x \mapsto (-x_1, x_2, \dots, x_N)$ ) are not elements of the symmetry group and thus the  $+/-$  symmetry of  $T_i^2$  can not be used. To start on the construction problem we will determine all the harmonic homogeneous polynomials annihilated by  $T_j$  for  $2 < j \leq N$ . They are the analogues of ordinary harmonic polynomials in two variables and thus we call them *planar*. In this situation there is a natural symmetry based on the transposition  $(1, 2)$ : polynomials  $f$  satisfying  $f(x(1, 2)) = f(x)$  are called *symmetric* and those satisfying  $f(x(1, 2)) = -f(x)$  are called *antisymmetric*. Then  $T_1 + T_2$  preserves the symmetry type and  $T_1 - T_2$  reverses it. This property is relevant since  $T_1^2 + T_2^2 = \frac{1}{2}(T_1 + T_2)^2 + \frac{1}{2}(T_1 - T_2)^2$ .

Section 2 describes the basis of polynomials used in the construction, sets up and solves the recurrence equations required to produce symmetric and antisymmetric harmonic polynomials. Also the formulae for the actions of  $T_1 \pm T_2$  on the harmonics are derived. In Section 3 the inner product structures involving the weight function  $w_\kappa$  are defined and the structural constants for the harmonic polynomials are computed. By means of Clifford algebra techniques one can define an operator of Dirac type and Section

4 describes this theory and produces the planar monogenic polynomials. Finally Section 5 contains technical material providing proofs for some of the results appearing in Sections 2 and 3.

## 2 The $p$ -Basis and Construction of Harmonic Polynomials

The natural numbers  $\{0, 1, 2, 3, \dots\}$  are denoted by  $\mathbb{N}_0$ . The largest integer  $\leq t \in \mathbb{R}$  is denoted by  $[t]$ . Suppose  $f$  is a polynomial in  $x \in \mathbb{R}^N$  then  $(1, 2)f$  denotes the polynomial  $f(x(1, 2))$ . To facilitate working with generating functions we introduce the notation  $\text{coef}(f, g_j) := c_j$  for the designated coefficient of  $f$  in the expansion  $f = \sum_i c_i g_i$  in terms of a basis  $\{g_i\}$ . Throughout  $\kappa$  is a fixed parameter, implicit in  $\{T_i\}$ , generally subject to  $\kappa > -\frac{1}{N}$ .

The  $p$ -basis associated to the operators  $\{T_i\}$  is constructed as follows: for  $1 \leq i \leq N$  the polynomials  $p_n(x_i; x)$  are given by the generating function

$$\sum_{n=0}^{\infty} p_n(x_i; x) r^n = (1 - rx_i)^{-1} \prod_{j=1}^N (1 - rx_j)^{-\kappa};$$

then for  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$  (the multi-indices), define  $p_\alpha := \prod_{i=1}^N p_{\alpha_i}(x_i; x)$ . The set  $\{p_\alpha : \alpha \in \mathbb{N}_0^N\}$  is a basis for the polynomials for generic  $\kappa$ . The key property is that  $T_j p_n(x_i; x) = 0$  for  $j \neq i$ . From [4, Sect. 10.3] we find

$$\begin{aligned} T_i p_\alpha &= (N\kappa + \alpha_i) p_{\alpha_i-1}(x_i; x) \prod_{m \neq i} p_{\alpha_m}(x_m; x) \\ &+ \kappa \sum_{j \neq i} \sum_{m=0}^{\alpha_j-1} (p_{\alpha_i+\alpha_j-1-m}(x_i; x) p_m(x_j; x) - p_m(x_i; x) p_{\alpha_i+\alpha_j-1-m}(x_j; x)) \\ &\quad \times \prod_{n \neq i, j} p_{\alpha_n}(x_n; x), \end{aligned} \tag{1}$$

if  $\alpha_i > 0$ , and  $T_i p_\alpha = 0$  if  $\alpha_i = 0$ .

Set up a symbolic calculus by letting  $p_j^n$  denote  $p_n(x_j; x)$ ; more formally define a linear isomorphism from ordinary polynomials to polynomials in the variables  $\{p_1, \dots, p_N\}$ :

$$\Psi p_\alpha = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_N^{\alpha_N}, \quad \alpha \in \mathbb{N}_0^N,$$

extended by linearity. Thus  $\Psi \sum_{n=0}^{\infty} p_n(x_i; x) r^n = (1 - p_i r)^{-1}$ . In this form the action of  $T_i$  (technically  $\Psi T_i \Psi^{-1}$ ) on a function of  $(p_1, \dots, p_N)$  is given by

$$T_i f(p) = \frac{\partial f}{\partial p_i} + N\kappa \frac{f - (p_i \rightarrow 0) f}{p_i} + \kappa \sum_{j=1, j \neq i}^N \frac{(p_i \rightarrow p_j) f + (p_j \rightarrow p_i) f - f - (p_j \longleftrightarrow p_i) f}{p_i - p_j}.$$

The operators  $(p_i \rightarrow 0)$  and  $(p_i \rightarrow p_j)$  replace  $p_i$  by 0 and  $p_j$  respectively, while  $(p_j \longleftrightarrow p_i)$  is the transposition. It suffices to examine the effect of the formula on monomials  $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_N^{\alpha_N}$  and for  $i = 1$ . The first two terms produce  $(\alpha_1 + N\kappa)$  if  $\alpha_1 > 0$ , else 0. In the sum, the (typical) term for  $j = 2$  is  $(p_1^{\alpha_1 + \alpha_2} + p_2^{\alpha_1 + \alpha_2} - p_1^{\alpha_1} p_2^{\alpha_2} - p_1^{\alpha_2} p_2^{\alpha_1}) \prod_{m=3}^N p_m^{\alpha_m} / (p_1 - p_2)$ . A simple calculation shows this is the image under  $\Psi$  of the corresponding term in equation (1). This method was used in [3] to find planar harmonics of type  $B$  (the group generated by sign-changes and permutation of coordinates).

From here on we will be concerned with polynomials in  $p_1, p_2$ , that is, exactly the set of polynomials annihilated by  $T_j$  for  $2 < j \leq N$ . Set  $p_{i,j} := p_i(x_1; x) p_j(x_2; x)$  so that  $\Psi p_{i,j} = p_i^j p_2^j$ . For each degree  $\geq 1$  there are two independent harmonic polynomials, that is,  $(T_1^2 + T_2^2) f = 0$ , and a convenient orthogonal decomposition is by the action of  $(1, 2)$ ; symmetric:  $(1, 2) f = f$ , and antisymmetric:  $(1, 2) f = -f$ , to be designated by  $+$  and  $-$  superscripts, respectively. We use the operators  $T_1 + T_2$  and  $T_1 - T_2$  (note  $(T_1^2 + T_2^2) = \frac{1}{2} (T_1 + T_2)^2 + \frac{1}{2} (T_1 - T_2)^2$ ). The harmonic polynomials will be expressed in the basis functions (symmetric)  $\phi_{nj}$  and (antisymmetric)  $\psi_{nj}$  with generating functions  $u_1, u_2$  (and  $s := \frac{1}{2} (z + z^{-1})$ ) given by

$$\begin{aligned} w_1 &:= (1 - ztp_1)^{-1} (1 - z^{-1}tp_2)^{-1}, \\ w_2 &:= (1 - z^{-1}tp_1)^{-1} (1 - ztp_2)^{-1}, \\ u_1 &:= \frac{1}{2} (w_1 + w_2) = \frac{1 - st(p_1 + p_2) + t^2 p_1 p_2}{(1 - 2stp_1 + t^2 p_1^2)(1 - 2stp_2 + t^2 p_2^2)}, \\ u_2 &:= \left( z - \frac{1}{z} \right)^{-1} (w_1 - w_2) = \frac{t(p_1 - p_2)}{(1 - 2stp_1 + t^2 p_1^2)(1 - 2stp_2 + t^2 p_2^2)}, \end{aligned}$$

$$u_1 = \sum_{n=0}^{\infty} t^n \sum_{j=0}^n s^j \phi_{nj},$$

$$u_2 = \sum_{n=1}^{\infty} t^n \sum_{j=0}^n s^j \psi_{nj}.$$

There are parity conditions:  $\phi_{nj} \neq 0$  implies  $j \equiv n \pmod{2}$  and  $\psi_{nj} \neq 0$  implies  $j \equiv (n-1) \pmod{2}$ . These are formal power series and convergence is not important (but is assured if  $\max(|zt|, |t/z|) < (\max_i |x_i|)^{-1}$ ).

The following expressions are derived in Section 5: for  $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$

$$\begin{aligned} \phi_{n,n-2j} &= 2^{n-1-2j} \sum_{i=0}^j \frac{(n+1-2j)_{2i}}{i! (1-n+2j-2i)_i} (p_{n-j+i,j-i} + p_{j-i,n-j+i}) \\ &= 2^{n-1-2j} (p_{n-j,j} + p_{j,n-j}) + 2^{n-1-2j} \\ &\quad \times \sum_{i=1}^j (n-2j+2i) \frac{(n+1-2j)_{i-1}}{i!} (-1)^i (p_{n-j+i,j-i} + p_{j-i,n-j+i}); \end{aligned}$$

and for  $0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$

$$\psi_{n,n-1-2j} = 2^{n-1-2j} \sum_{i=0}^j \frac{(n-2j)_i}{i!} (-1)^i (p_{n-j+i,j-i} - p_{j-i,n-j+i}).$$

The reason for the use of this basis is that the actions of  $T_1 + T_2$  and  $T_1 - T_2$  have relatively simple expressions. It is easy to verify that (set  $\partial_v := \frac{\partial}{\partial v}$  for a variable  $v$ )

$$\begin{aligned} \partial_{p_1} w_1 &= zt(w_1 + \frac{t}{2} \partial_t w_1) + \frac{z^2 t}{2} \partial_z w_1, \quad \partial_{p_2} w_1 = \frac{t}{z} (w_1 + \frac{t}{2} \partial_t w_1) - \frac{t}{2} \partial_z w_1 \\ \partial_{p_1} w_2 &= \frac{t}{z} (w_2 + \frac{t}{2} \partial_t w_2) - \frac{t}{2} \partial_z w_2, \quad \partial_{p_2} w_2 = zt(w_2 + \frac{t}{2} \partial_t w_2) + \frac{z^2 t}{2} \partial_z w_2. \end{aligned}$$

After some calculations involving  $\frac{\partial}{\partial z} = \frac{1}{2} (1 - z^{-2}) \frac{\partial}{\partial s}$  we obtain

$$\begin{aligned} (\partial_{p_1} + \partial_{p_2}) u_1 &= 2stu_1 + st^2 \partial_t u_1 + (s^2 - 1) t \partial_s u_1, \\ (\partial_{p_1} - \partial_{p_2}) u_1 &= (3s^2 - 2) tu_2 + (s^2 - 1) t^2 \partial_t u_2 + s(s^2 - 1) t \partial_s u_2, \\ (\partial_{p_1} + \partial_{p_2}) u_2 &= 3stu_2 + st^2 \partial_t u_2 + (s^2 - 1) t \partial_s u_2, \\ (\partial_{p_1} - \partial_{p_2}) u_2 &= 2tu_1 + t^2 \partial_t u_1 + st \partial_s u_1. \end{aligned}$$

Applying

$$\begin{aligned}
T_1 + T_2 - \partial_{p_1} - \partial_{p_2} &= N\kappa \frac{1 - (p_1 \rightarrow 0)}{p_1} + N\kappa \frac{1 - (p_2 \rightarrow 0)}{p_2}, \\
T_1 - T_2 - \partial_{p_1} + \partial_{p_2} &= N\kappa \frac{1 - (p_1 \rightarrow 0)}{p_1} - N\kappa \frac{1 - (p_2 \rightarrow 0)}{p_2} \\
&\quad + 2\kappa \frac{(p_1 \rightarrow p_2) + (p_2 \rightarrow p_1) - 1 - (p_1 \longleftrightarrow p_2)}{p_1 - p_2}
\end{aligned}$$

to  $u_1$  and  $u_2$  yields

$$\begin{aligned}
(T_1 + T_2 - \partial_{p_1} - \partial_{p_2}) u_1 &= 2stN\kappa u_1, \\
(T_1 - T_2 - \partial_{p_1} + \partial_{p_2}) u_1 &= 2t^2 (N(s^2 - 1) + 1) \kappa u_2, \\
(T_1 + T_2 - \partial_{p_1} - \partial_{p_2}) u_2 &= 2tsN\kappa u_2, \\
(T_1 - T_2 - \partial_{p_1} + \partial_{p_2}) u_2 &= 2tN\kappa u_1.
\end{aligned}$$

Applying these to the generating functions results in

$$(T_1 + T_2) \phi_{nj} = -(j+1) \phi_{n-1,j+1} + (2N\kappa + n + j) \phi_{n-1,j-1}, \quad (2)$$

$$(T_1 - T_2) \phi_{nj} = -(2N\kappa - 2\kappa + n + j + 1) \psi_{n-1,j} + (2N\kappa + n + j) \psi_{n-1,j-2}, \quad (3)$$

$$(T_1 + T_2) \psi_{nj} = -(j+1) \psi_{n-1,j+1} + (2N\kappa + n + j + 1) \psi_{n-1,j-1}, \quad (4)$$

$$(T_1 - T_2) \psi_{nj} = (2N\kappa + n + j + 1) \phi_{n-1,j}. \quad (5)$$

We will state the expressions for the harmonic polynomials before their derivations, however it is necessary to define two families of polynomials via three-term relations. The motivation comes later.

**Definition 2** For  $n = 0, 1, 2, \dots$  define two families of monic polynomials by

$$\begin{aligned}
g_0^o(v) &= 1, g_{n+1}^o(v) = (v + 3n + 1) g_n^o(v) - n(2n - 1) g_{n-1}^o(v), \\
g_0^e(v) &= 1, g_{n+1}^e(v) = (v + 3n + 2) g_n^e(v) - n(2n + 1) g_{n-1}^e(v).
\end{aligned}$$

The first few polynomials are

$$\begin{aligned}
g_1^o(v) &= v + 1, \\
g_2^o(v) &= v^2 + 5v + 3, \\
g_3^o(v) &= v^3 + 12v^2 + 32v + 15
\end{aligned}$$

and

$$\begin{aligned} g_1^e(v) &= v + 2, \\ g_2^e(v) &= v^2 + 7v + 7, \\ g_3^e(v) &= v^3 + 15v^2 + 53v + 36. \end{aligned}$$

**Definition 3** For  $m = 0, 1, 2, \dots$  let

$$\begin{aligned} h_{2m+1}^- &:= \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} \psi_{2m+1, 2j}, \\ h_{2m}^- &:= \sum_{j=0}^{m-1} 2^{-j} \frac{g_j^e(N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} \psi_{2m, 2j+1}, \\ h_{2m+1}^+ &:= \sum_{j=0}^m 2^{-j} \frac{g_j^e(N\kappa - \kappa + m + 1)}{(N\kappa + m + 2)_j} \phi_{2m+1, 2j+1} \\ h_{2m}^+ &:= \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 1)_j} \phi_{2m, 2j}, . \end{aligned}$$

First we show that the antisymmetric polynomials  $h_n^-$  are harmonic. We will use the last relation to produce symmetric harmonic polynomials from the antisymmetric ones. Combining equations (3), (5), (4) obtain

$$\begin{aligned} &\left( (T_1 + T_2)^2 + (T_1 - T_2)^2 \right) \psi_{nj} \\ &= (j+1)(j+2) \psi_{n-2, j+2} - (2N\kappa + n + j + 1)(2N\kappa - 2\kappa + n + 3j + 1) \psi_{n-2, j} \\ &\quad + 2(2N\kappa + n + j + 1)(2N\kappa + n - j + 1) \psi_{n-2, j-2}. \end{aligned}$$

Suppose  $h_n^- = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_{n-1-2j} \psi_{n, n-1-2j}$  is harmonic then the coefficient of  $\psi_{n-2, n-2j-1}$  in  $2\Delta_\kappa h_n^-$  is

$$\begin{aligned} 0 &= 8(N\kappa + n - j + 1)(N\kappa + n - j) c_{n+1-2j} \\ &\quad - 4(N\kappa - \kappa + 2n - 1 - 3j)(N\kappa + n - j) c_{n-1-2j} \\ &\quad + (n - 2j - 2)(n - 2j - 1) c_{n-3-2j} \end{aligned}$$

The range of  $j$  is derived from the inequality  $0 \leq n - 2j - 1 \leq n - 3$ , that is  $1 \leq j \leq \frac{n-1}{2}$ . Two sets of formulae arise depending on the parity of  $n$ . The equations are considered as recurrences.

Suppose  $n = 2m + 1$  then the starting point is for  $j = m$

$$8(N\kappa + m + 2)(N\kappa + m + 1) c_2 - 4(N\kappa - \kappa + m + 1)(N\kappa + m + 1) c_0 = 0,$$

thus

$$c_2 = \frac{1}{2} \frac{N\kappa - \kappa + m + 1}{N\kappa + m + 2} c_0.$$

Set  $j = m - i$  to obtain

$$\begin{aligned} & c_{2i+2} \\ &= \frac{1}{2} \frac{(N\kappa - \kappa + m + 1 + 3i)}{N\kappa + m + 2 + i} c_{2i} - \frac{1}{4} \frac{(2i - 1)i}{(N\kappa + m + 2 + i)(N\kappa + m + 1 + i)} c_{2i-2}. \end{aligned}$$

To simplify the recurrences let  $\gamma_i^o = 2^i c_{2i} (N\kappa + m + 2)_i / c_0$  then  $\gamma_0^o = 1$  and

$$\gamma_{i+1}^o = (N\kappa - \kappa + m + 1 + 3i) \gamma_i^o - i(2i - 1) \gamma_{i-1}^o.$$

which agrees with the recurrence for  $g_i^o$  with  $v = N\kappa - \kappa + m$ .

Thus the antisymmetric harmonic polynomial of degree  $2m + 1$  (normalized by  $c_0 = 1$ ) is

$$h_{2m+1}^- = \sum_{j=0}^m 2^{-j} \frac{g_j^o (N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} \psi_{2m+1, 2j}.$$

Suppose  $n = 2m$  then the starting point is for  $j = m - 1$

$$8(N\kappa + m + 2)(N\kappa + m + 1)c_3 - 4(N\kappa - \kappa + m + 2)(N\kappa + m + 1)c_1 = 0,$$

so that

$$c_3 = \frac{1}{2} \frac{N\kappa - \kappa + m + 2}{N\kappa + m + 2} c_1.$$

Set  $j = m - 1 - i$  to obtain

$$\begin{aligned} & c_{2i+3} \\ &= \frac{1}{2} \frac{(N\kappa - \kappa + m + 2 + 3i)}{N\kappa + m + 2 + i} c_{2i+1} - \frac{1}{4} \frac{(2i + 1)i}{(N\kappa + m + 2 + i)(N\kappa + m + 1 + i)} c_{2i-1}. \end{aligned}$$

Similarly to the previous calculation let  $\gamma_i^e := 2^i c_{2i+1} (N\kappa + m + 2)_i / c_1$  then  $\gamma_0^e = 1$  and

$$\gamma_{i+1}^e = (N\kappa - \kappa + m + 2 + 3i) \gamma_i^e - i(2i + 1) \gamma_{i-1}^e.$$

This agrees with the recurrence for  $g_i^e$  with  $v = N\kappa - \kappa + m$ . Thus the antisymmetric harmonic polynomial of degree  $2m$  (normalized by  $c_1 = 1$ ) is

$$h_{2m}^- = \sum_{j=0}^{m-1} 2^{-j} \frac{g_j^e (N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} \psi_{2m, 2j+1}.$$



Applying  $(T_1 - T_2)$  to a harmonic polynomial clearly produces another harmonic polynomial; thus by equation (5)

$$\begin{aligned}
& \frac{1}{2(N\kappa + m + 1)} (T_1 - T_2) h_{2m+1}^- \\
&= \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 1)_{j+1}} (N\kappa + m + j + 1) \phi_{2m,2j} \\
&= \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 1)_j} \phi_{2m,2j} = h_{2m}^+
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2(N\kappa + m + 1)} (T_1 - T_2) h_{2m}^- \\
&= \sum_{j=0}^{m-1} 2^{-j} \frac{g_j^e(N\kappa - \kappa + m)}{(N\kappa + m + 1)_{j+1}} (N\kappa + m + j + 1) \phi_{2m-1,2j+1} \\
&= \sum_{j=0}^{m-1} 2^{-j} \frac{g_j^e(N\kappa - \kappa + m)}{(N\kappa + m + 1)_j} \phi_{2m-1,2j+1} = h_{2m-1}^+.
\end{aligned}$$

We have proven:

**Proposition 4** *The polynomials  $h_n^+$  and  $h_n^-$  are harmonic.*

For use in the sequel we find expressions for  $T_1 \pm T_2$  applied to  $h_n^+$  and  $h_n^-$ .

**Proposition 5** *The actions of  $T_1 \pm T_2$  on the antisymmetric polynomials  $h_n^-$  are*

$$\begin{aligned}
(T_1 - T_2) h_{2m+1}^- &= 2(N\kappa + m + 1) h_{2m}^+, \\
(T_1 - T_2) h_{2m}^- &= 2(N\kappa + m + 1) h_{2m-1}^+, \\
(T_1 + T_2) h_{2m+1}^- &= (N\kappa - \kappa + m) h_{2m}^-, \\
(T_1 + T_2) h_{2m}^- &= 2(N\kappa + m + 1) h_{2m-1}^-,
\end{aligned}$$

and the actions on the symmetric polynomials  $h_n^+$  are

$$\begin{aligned}
(T_1 - T_2) h_{2m+1}^+ &= -(N\kappa - \kappa + m) h_{2m}^-, \\
(T_1 - T_2) h_{2m}^+ &= -(N\kappa - \kappa + m) h_{2m-1}^-, \\
(T_1 + T_2) h_{2m+1}^+ &= 2(N\kappa + m + 1) h_{2m}^+, \\
(T_1 + T_2) h_{2m}^+ &= (N\kappa - \kappa + m) h_{2m-1}^+.
\end{aligned}$$

Since the resulting polynomials are harmonic it suffices to consider just one term in their expansions. The coefficients of the lowest index term ( $\phi_{2m-1,1}, \phi_{2m,0}, \psi_{2m-1,0}, \psi_{2m,1}$  for  $h_{2m-1}^+, h_{2m}^+, h_{2m-1}^-, h_{2m}^-$  respectively) on the right sides arise from at most two terms on the left. The details are in Section 5.

### 3 Inner Products and Structure Constants

Let  $\mu$  denote the Gaussian measure  $(2\pi)^{-N/2} e^{-|x|^2/2} dx$  on  $\mathbb{R}^N$ , (where  $dx$  is the Lebesgue measure), and let  $m$  denote the normalized surface measure on  $S_{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ . The weight function  $w_\kappa(x) := \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\kappa}$ . The constants  $c_\kappa$  and  $c'_\kappa$  are defined by  $c_\kappa \int_{\mathbb{R}^N} w_\kappa d\mu = 1$  and  $c'_\kappa \int_{S_{N-1}} w_\kappa dm = 1$ . It is known (Macdonald-Mehta-Selberg integral) that  $c_\kappa = \prod_{j=2}^N \left( \frac{\Gamma(\kappa+1)}{\Gamma(j\kappa+1)} \right)$ . There are three inner products for polynomials associated with  $\Delta_\kappa$ . For polynomials  $f, g$  define

1.  $\langle f, g \rangle_\kappa := f(T_1, \dots, T_N) g(x) |_{x=0}$  (evaluated at  $x = 0$ );
2.  $\langle f, g \rangle_G := c_\kappa \int_{\mathbb{R}^N} f g w_\kappa d\mu$ , the Gaussian inner product;
3.  $\langle f, g \rangle_S := c'_\kappa \int_{S_{N-1}} f g w_\kappa dm$ .

The details can be found in [4, Ch. 7.2]. There are important relations among them:  $\langle f, g \rangle_\kappa = \langle e^{-\Delta_\kappa/2} f, e^{-\Delta_\kappa/2} g \rangle_G$  (note that the series  $\sum_{j \geq 0} \frac{1}{j!} \left(-\frac{\Delta_\kappa}{2}\right)^j f$  terminates for any polynomial  $f$ ) and if  $f$  is homogeneous of degree  $2n$  then

$$\int_{\mathbb{R}^N} f w_\kappa d\mu = 2^{n+N(N-1)\kappa/2} \frac{\Gamma\left(\frac{N}{2}((N-1)\kappa+1)+n\right)}{\Gamma\left(\frac{N}{2}\right)} \int_{S_{N-1}} f w_\kappa dm.$$

Specialized to  $f = 1$  this shows that

$$c'_\kappa = 2^{N(N-1)\kappa/2} \frac{\Gamma\left(\frac{N}{2}((N-1)\kappa+1)\right)}{\Gamma\left(\frac{N}{2}\right)} c_\kappa$$

and thus

$$c_\kappa \int_{\mathbb{R}^N} f w_\kappa d\mu = 2^{N(N-1)\kappa/2} \left(\frac{N}{2}((N-1)\kappa+1)\right)_n c'_\kappa \int_{S_{N-1}} f w_\kappa dm.$$

As a consequence if  $f$  and  $g$  are harmonic and homogeneous of degrees  $m, n$  respectively then

$$\langle f, g \rangle_\kappa = \langle f, g \rangle_G = 2^n \left( \frac{N}{2} ((N-1)\kappa + 1) \right)_n \delta_{mn} \langle f, g \rangle_S. \quad (6)$$

It is a fundamental result that  $\deg f \neq \deg g$  implies  $\langle f, g \rangle_S = 0$ .

To find  $\langle f, f \rangle_\kappa$  for the harmonic polynomials  $h_n^+, h_n^-$  we will need the values of  $\phi_{nj}$  and  $\psi_{nj}$  at  $x = (x_1, x_2, 0, \dots, 0)$ . In terms of the generating functions

$$\Psi^{-1} (1 - rp_1)^{-1} = \Psi^{-1} \sum_{n=0}^{\infty} p_1^n r^n = (1 - rx_1)^{-\kappa-1} (1 - rx_2)^{-\kappa},$$

thus

$$\begin{aligned} w_1(x) &:= (1 - ztx_1)^{-1-\kappa} (1 - ztx_2)^{-\kappa} (1 - z^{-1}tx_2)^{-1-\kappa} (1 - z^{-1}tx_1)^{-\kappa}, \\ w_2(x) &:= (1 - ztx_2)^{-1-\kappa} (1 - ztx_1)^{-\kappa} (1 - z^{-1}tx_1)^{-1-\kappa} (1 - z^{-1}tx_2)^{-\kappa}, \end{aligned}$$

and

$$\begin{aligned} u_1(x) &= \frac{1}{2} (w_1(x) + w_2(x)) \\ &= \frac{(1 - ztx_2)(1 - z^{-1}tx_1) + (1 - ztx_1)(1 - z^{-1}tx_2)}{2 \{ (1 - 2stx_1 + x_1^2 t^2) (1 - 2stx_2 + x_2^2 t^2) \}^{\kappa+1}} \\ &= \frac{1 - (x_1 + x_2)st + x_1 x_2 t^2}{\{ (1 - 2stx_1 + x_1^2 t^2) (1 - 2stx_2 + x_2^2 t^2) \}^{\kappa+1}}, \\ u_2(x) &= \left( z - \frac{1}{z} \right)^{-1} (w_1(x) - w_2(x)) \\ &= \frac{(x_1 - x_2)t}{\{ (1 - 2stx_1 + x_1^2 t^2) (1 - 2stx_2 + x_2^2 t^2) \}^{\kappa+1}}, \end{aligned}$$

because  $(z - z^{-1})^{-1} \{ (1 - ztx_2)(1 - z^{-1}tx_1) - (1 - ztx_1)(1 - z^{-1}tx_2) \} = (x_1 - x_2)t$ . Thus  $\phi_{nj}(x) = \text{coef}(u_1(x), t^n s^j)$  and  $\psi_{nj}(x) = \text{coef}(u_2(x), t^n s^j)$ .

By the (1, 2) symmetry (both  $w_\kappa$  and  $h_n^+$  are invariant and  $h_n^-$  changes sign) the inner products  $\langle h_n^+, h_n^- \rangle = 0$ . Next we compute the pairing  $\langle f, f \rangle_\kappa$  for the harmonic polynomials. Since they are annihilated by  $T_j$  for  $j > 2$  these values are given by  $f(T_1, T_2, 0, \dots) f$ . We use the harmonicity of  $f$ , that is,  $(T_1 - T_2)^2 f = -(T_1 + T_2)^2 f$ . The same relation holds when  $f$  is replaced

by  $q(T_1, T_2) f$  for any polynomial  $q$ . Suppose  $\deg f = n$  and express

$$f(x_1, x_2, 0 \dots) = \sum_{j=0}^n c_j (x_1 + x_2)^{n-j} (x_1 - x_2)^j.$$

If  $(1, 2) f = f$  then  $c_j = 0$  for odd  $j$  and

$$\begin{aligned} f(T_1, T_2, 0 \dots) f &= \sum_{j=0}^{\lfloor n/2 \rfloor} c_{2j} (T_1 + T_2)^{n-2j} (T_1 - T_2)^{2j} f \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} c_{2j} (-1)^j (T_1 + T_2)^n f. \end{aligned}$$

Set  $x_1 = 1 + i, x_2 = 1 - i$  then

$$f(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} c_{2j} 2^{n-2j} (2i)^{2j} = 2^n \sum_{j=0}^{\lfloor n/2 \rfloor} c_{2j} (-1)^j;$$

and thus

$$f(T_1, T_2, 0 \dots) f = 2^{-n} f(1 + i, 1 - i, 0 \dots) (T_1 + T_2)^n f.$$

Proceeding similarly for  $(1, 2) f = -f$  where  $c_j = 0$  for even  $j$  we obtain

$$\begin{aligned} f(T_1, T_2, 0 \dots) f &= \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_{2j+1} (T_1 + T_2)^{n-1-2j} (T_1 - T_2)^{2j+1} f \\ &= \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_{2j+1} (-1)^j (T_1 - T_2) (T_1 + T_2)^{n-1} f, \end{aligned}$$

and

$$f(1 + i, 1 - i, 0 \dots) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_{2j+1} 2^{n-1-2j} (2i)^{2j+1} = i 2^n \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} c_{2j+1} (-1)^j.$$

Thus

$$f(T_1, T_2, 0 \dots) f = -i 2^{-n} f(1 + i, 1 - i, 0 \dots) (T_1 + T_2)^{n-1} (T_1 - T_2) f.$$

First the symmetric case (by Proposition 5):

$$(T_1 + T_2)^2 h_{2m}^+ = (T_1 + T_2) (N\kappa - \kappa + m) h_{2m-1}^+ = 2(N\kappa - \kappa + m) (N\kappa + m) h_{2m-2}^+,$$

and it follows by induction that

$$\begin{aligned}(T_1 + T_2)^{2m} h_{2m}^+ &= 2^m (N\kappa - \kappa + 1)_m (N\kappa + 1)_m, \\ (T_1 + T_2)^{2m+1} h_{2m+1}^+ &= 2(N\kappa + m + 1)(T_1 + T_2)^{2m} h_{2m}^+ \\ &= 2^{m+1} (N\kappa - \kappa + 1)_m (N\kappa + 1)_{m+1}.\end{aligned}$$

For the antisymmetric case

$$\begin{aligned}(T_1 + T_2)^{2m-1} (T_1 - T_2) h_{2m}^- &= 2(N\kappa + m + 1)(T_1 + T_2)^{2m-1} h_{2m-1}^+ \\ &= 2^{m+1} (N\kappa + m + 1)(N\kappa - \kappa + 1)_{m-1} (N\kappa + 1)_m \\ &= 2^{m+1} (N\kappa - \kappa + 1)_{m-1} (N\kappa + 1)_{m+1},\end{aligned}$$

and

$$\begin{aligned}(T_1 + T_2)^{2m} (T_1 - T_2) h_{2m+1}^- &= 2(N\kappa + m + 1)(T_1 + T_2)^{2m} h_{2m}^+ \\ &= 2^{m+1} (N\kappa + m + 1)(N\kappa - \kappa + 1)_m (N\kappa + 1)_m \\ &= 2^{m+1} (N\kappa - \kappa + 1)_m (N\kappa + 1)_{m+1}.\end{aligned}$$

The values  $\phi_{nj}(1+i, 1-i, 0\dots)$  and  $\psi_{nj}(1+i, 1-i, 0\dots)$  are found by computing the generating functions:

$$u_1(1+i, 1-i, 0\dots) = \frac{1 - 2st + 2t^2}{(1 - 4st + 8s^2t^2 - 8st^3 + 4t^4)^{\kappa+1}}$$

(the term in the denominator is  $(1 - 2st + 2t^2)^2 - 4(1 - s^2)t^2$ ) and

$$u_2(1+i, 1-i, 0\dots) = \frac{2it}{(1 - 4st + 8s^2t^2 - 8st^3 + 4t^4)^{\kappa+1}}.$$

**Definition 6** For  $n = 0, 1, 2, \dots$ ,  $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$  and parameters  $\alpha, \beta$  let

$$\begin{aligned}S(n, j; \alpha, \beta) &:= \sum_{\ell=0}^{\lfloor n/2 \rfloor} \sum_{i=\max(0, \ell+j-\lfloor n/2 \rfloor)}^{\min(\ell, j)} \frac{(\alpha+1)_\ell (2\alpha + \beta + 2\ell)_{n-2\ell-j+i}}{i! (\ell-i)! (j-i)! (n-2\ell-2j+2i)!} (-1)^{\ell+j} 2^{n-j+i}.\end{aligned}$$

**Proposition 7** For  $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$

$$\begin{aligned}\phi_{n,n-2j}(1+i, 1-i, 0, \dots) &= S(n, j; \kappa, 1), \\ \psi_{n+1,n-2j}(1+i, 1-i, 0, \dots) &= 2iS(n, j; \kappa, 2).\end{aligned}$$

The proof is in Proposition 8.

Thus

$$\begin{aligned}\langle h_{2m}^+, h_{2m}^+ \rangle_\kappa &= 2^{-2m} h_{2m}^+ (1+i, 1-i, 0, \dots) (T_1 + T_2)^{2m} h_{2m}^+ \\ &= 2^{-m} \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 1)_j} S(2m, m-j; \kappa, 1) \\ &\quad \times (N\kappa - \kappa + 1)_m (N\kappa + 1)_m, \\ \langle h_{2m+1}^+, h_{2m+1}^+ \rangle_\kappa &= 2^{-2m-1} h_{2m+1}^+ (1+i, 1-i, 0, \dots) (T_1 + T_2)^{2m+1} h_{2m+1}^+ \\ &= 2^{-m} \sum_{j=0}^m 2^{-j} \frac{g_j^e(N\kappa - \kappa + m + 1)}{(N\kappa + m + 2)_j} S(2m+1, m-j; \kappa, 1) \\ &\quad \times (N\kappa - \kappa + 1)_m (N\kappa + 1)_{m+1},\end{aligned}$$

and

$$\begin{aligned}\langle h_{2m}^-, h_{2m}^- \rangle_\kappa &= -i2^{-2m} h_{2m}^- (1+i, 1-i, 0, \dots) (T_1 + T_2)^{2m-1} (T_1 - T_2) h_{2m}^- \\ &= 2^{-m+2} \sum_{j=0}^{m-1} 2^{-j} \frac{g_j^e(N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} S(2m, m-j-1; \kappa, 2) \\ &\quad \times (N\kappa - \kappa + 1)_{m-1} (N\kappa + 1)_{m+1}, \\ \langle h_{2m+1}^-, h_{2m+1}^- \rangle_\kappa &= -i2^{-2m} h_{2m+1}^- (1+i, 1-i, 0, \dots) (T_1 + T_2)^{2m} (T_1 - T_2) h_{2m+1}^- \\ &= 2^{2-m} \sum_{j=0}^m 2^{-j} \frac{g_j^o(N\kappa - \kappa + m)}{(N\kappa + m + 2)_j} S(2m+1, m-j; \kappa, 2) \\ &\quad \times (N\kappa - \kappa + 1)_m (N\kappa + 1)_{m+1}.\end{aligned}$$

The values of  $\langle h_n^+, h_n^+ \rangle_S$  and  $\langle h_n^-, h_n^- \rangle_S$  can now be found by equation (6). The expressions are complicated; due to the fact that sign-changes are not in the symmetry group.

## 4 The Dirac Operator and Monogenic Polynomials

We use the Clifford algebra  $C\ell_N$  over  $\mathbb{R}$  generated by  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_N\}$  with relations  $\mathbf{e}_i^2 = -1$  (that is, negative signature) and  $\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i$  for

$i \neq j$ . The type- $A$  Dirac operator acting on polynomials in  $x \in \mathbb{R}^N$  with coefficients in  $C\ell_N$  is defined by

$$Df := \sum_{i=1}^N \mathbf{e}_i T_i f;$$

this implies  $D^2 = -\sum_{i=1}^N T_i^2 = -\Delta_\kappa$ . A polynomial  $f$  is said to be monogenic if  $Df = 0$ . The situation where the underlying symmetry group is  $\mathbb{Z}_2^N$  has been investigated by De Bie, Genest and Vinet [1],[2]. The planar harmonic polynomials found in the previous sections can be used to construct monogenic polynomials. They are of the form  $f_n = h_n^+ + \varepsilon h_n^-$  with  $\varepsilon \in C\ell_N$ . By construction  $T_i f = 0$  for all  $i > 2$ . To fit with the formulae in Proposition 5 write  $\mathbf{e}_1 T_1 + \mathbf{e}_2 T_2 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)(T_1 + T_2) + \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2)(T_1 - T_2)$ . Even and odd  $n$  are handled separately.

$$\begin{aligned} & (\mathbf{e}_1 T_1 + \mathbf{e}_2 T_2) (h_{2m+1}^+ + \varepsilon h_{2m+1}^-) \\ &= (\mathbf{e}_1 + \mathbf{e}_2) (N\kappa + m + 1) h_{2m}^+ - \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2) (N\kappa - \kappa + m) h_{2m}^- \\ &+ (\mathbf{e}_1 - \mathbf{e}_2) \varepsilon (N\kappa + m + 1) h_{2m}^+ + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \varepsilon (N\kappa - \kappa + m) h_{2m}^-; \end{aligned}$$

the coefficients of  $(N\kappa + m + 1) h_{2m}^+$  and  $\frac{1}{2}(N\kappa - \kappa + m) h_{2m}^-$  are  $(\mathbf{e}_1 + \mathbf{e}_2) + (\mathbf{e}_1 - \mathbf{e}_2) \varepsilon$  and  $-(\mathbf{e}_1 - \mathbf{e}_2) + (\mathbf{e}_1 + \mathbf{e}_2) \varepsilon$  respectively. Both of these vanish for  $\varepsilon = \mathbf{e}_1 \mathbf{e}_2$ . Thus  $D(h_{2m+1}^+ + \mathbf{e}_1 \mathbf{e}_2 h_{2m+1}^-) = 0$ . Since  $D$  commutes with  $T_1 + T_2$  the polynomial  $(T_1 + T_2)(h_{2m+1}^+ + \mathbf{e}_1 \mathbf{e}_2 h_{2m+1}^-)$  is also monogenic, and  $(T_1 + T_2)(h_{2m+1}^+ + \mathbf{e}_1 \mathbf{e}_2 h_{2m+1}^-) = 2(N\kappa + m + 1) h_{2m}^+ + \mathbf{e}_1 \mathbf{e}_2 (N\kappa - \kappa + m) h_{2m}^-$ . This proves

$$\begin{aligned} & D(h_{2m+1}^+ + \mathbf{e}_1 \mathbf{e}_2 h_{2m+1}^-) = 0, \\ & D\left(h_{2m}^+ + \mathbf{e}_1 \mathbf{e}_2 \frac{N\kappa - \kappa + m}{2(N\kappa + m + 1)} h_{2m}^-\right) = 0. \end{aligned}$$

## 5 Derivations of Various Formulae

This section contains the derivations of some of the formulae appearing in the paper. The formulae for  $\phi_{n,j}$  and  $\psi_{n,j}$  are found by means of the

Chebyshev polynomials  $T_k$  and  $U_k$ .

$$\begin{aligned}
u_1 &= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} t^{k+m} p_1^k p_2^m (z^{k-m} + z^{m-k}) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} t^n \sum_{m=0}^n p_1^{n-m} p_2^m (z^{n-2m} + z^{2m-n}) \\
&= \sum_{n=0}^{\infty} t^n \sum_{m=0}^n p_1^{n-m} p_2^m \cos((n-2m)\theta) \\
&= \sum_{n=0}^{\infty} t^n \sum_{m=0}^n p_1^{n-m} p_2^m T_{|n-2m|}(s),
\end{aligned}$$

where  $z$  is replaced by  $e^{i\theta}$  and thus  $s = \cos \theta$ . The last inner sum can be written as  $\sum_{m=0}^{\lfloor n/2 \rfloor} \varepsilon_{n,m} (p_1^{n-m} p_2^m + p_1^m p_2^{n-m}) T_{n-2m}(s)$  where  $\varepsilon_{n,m} = 1$  except  $\varepsilon_{2m,m} = \frac{1}{2}$ . Then use the expansion  $T_k(s) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-k)_{2j}}{j!(1-k)_j} 2^{k-1-2j} s^{k-2j}$  for  $k \geq 1$  and extract the coefficient of  $s^{n-2j}$  to determine  $\phi_{n,n-2j}$ . Applying the same technique to  $u_2$  we obtain

$$\begin{aligned}
u_2 &= \frac{1}{z - z^{-1}} \sum_{n=0}^{\infty} t^n \sum_{m=0}^n p_1^{n-m} p_2^m (z^{n-2m} - z^{2m-n}) \\
&= \sum_{n=0}^{\infty} t^n \sum_{m=0}^n p_1^{n-m} p_2^m \frac{\sin((n-2m)\theta)}{\sin \theta} \\
&= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{\lfloor n/2 \rfloor} (p_1^{n-m} p_2^m - p_1^m p_2^{n-m}) \frac{\sin((n-2m)\theta)}{\sin \theta} \\
&= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{\lfloor n/2 \rfloor} (p_1^{n-m} p_2^m - p_1^m p_2^{n-m}) U_{n-1-2m}(s).
\end{aligned}$$

Then extract the coefficient of  $s^{n-1-2j}$  by means of the expansion  $U_k(s) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-k)_{2j}}{j!(-k)_j} 2^{k-2j} s^{k-2j}$  for  $k > 0$  to find  $\psi_{n,n-1-2j}$ .

**Proof.** (of Proposition 5). The formulae for  $(T_1 - T_2) h_n^-$  have already been proven. For  $(T_1 + T_2) h_n^-$  substitute  $n = 2m$  and  $j = 1$  in (4) to obtain  $\text{coef}((T_1 + T_2) h_{2m}^-, \psi_{2m-1,0}) = (2N\kappa + 2m + 2)$  and thus  $(T_1 + T_2) h_{2m}^- = 2(N\kappa + m + 1) h_{2m-1}^-$ , next substitute  $n = 2m + 1$  and  $j = 0, 2$  in (4) to



show that

$$\begin{aligned}
& \text{coef}((T_1 + T_2) h_{2m+1}^-, \psi_{2m,1}) \\
&= -\text{coef}(h_{2m+1}^-, \psi_{2m+1,0}) + (2N\kappa + 2m + 4) \text{coef}(h_{2m+1}^-, \psi_{2m+1,2}) \\
&= -1 + (N\kappa + m + 2) \frac{g_1^o(N\kappa - \kappa + m)}{N\kappa + m + 2} = N\kappa - \kappa + m.
\end{aligned}$$

For  $(T_1 + T_2) h_n^+$  substitute  $n = 2m + 1, j = 1$  in (2) to show

$$\begin{aligned}
\text{coef}((T_1 + T_2) h_{2m+1}^+, \phi_{2m,0}) &= (2N\kappa + 2m + 2) \text{coef}(h_{2m+1}^+, \phi_{2m+1,1}) \\
&= 2(N\kappa + m + 1);
\end{aligned}$$

next substitute  $n = 2m, j = 0, 2$  in (2) to show that

$$\begin{aligned}
& \text{coef}((T_1 + T_2) h_{2m}^{\pm}, \phi_{2m-1,1}) \\
&= -\text{coef}(h_{2m}^+, \phi_{2m,0}) + (2N\kappa + 2m + 2) \text{coef}(h_{2m}^+, \phi_{2m,2}) \\
&= -1 + (N\kappa + m + 1) \frac{g_1^o(N\kappa - \kappa + m)}{N\kappa + m + 1} = N\kappa - \kappa + m.
\end{aligned}$$

For  $(T_1 - T_2) h_n^+$  substitute  $n = 2m + 1, j = 1, 3$  in (3) to show

$$\begin{aligned}
& \text{coef}((T_1 - T_2) h_{2m+1}^+, \psi_{2m,1}) \\
&= -(2N\kappa - 2\kappa + 2m + 3) \text{coef}(h_{2m+1}^+, \phi_{2m+1,1}) \\
&+ (2N\kappa + 2m + 4) \text{coef}(h_{2m+1}^+, \phi_{2m+1,3}) \\
&= -(2N\kappa - 2\kappa + 2m + 3) + (N\kappa + m + 2) \frac{g_1^e(N\kappa - \kappa + m + 1)}{N\kappa + m + 2} \\
&= -(N\kappa - \kappa + m);
\end{aligned}$$

next substitute  $n = 2m, j = 0, 2$  in (3) to show

$$\begin{aligned}
& \text{coef}((T_1 - T_2) h_{2m}^+, \psi_{2m-1,0}) \\
&= -(2N\kappa - 2\kappa + 2m + 1) \text{coef}(h_{2m}^+, \phi_{2m,0}) + (2N\kappa + 2m + 2) \text{coef}(h_{2m}^+, \phi_{2m,2}) \\
&= -(2N\kappa - 2\kappa + 2m + 1) + (N\kappa + m + 1) \frac{g_1^o(N\kappa - \kappa + m)}{N\kappa + m + 1} \\
&= -(N\kappa - \kappa + m).
\end{aligned}$$

This completes the proof of Proposition 5. ■

To prove Proposition 7 note that the expressions for  $u_1$  and  $u_2/(2it)$  have the form

$$(1 - 2st + 2t^2)^{-2\kappa-\beta} \left( 1 - \frac{4(1-s^2)t^2}{(1-2st+2t^2)^2} \right)^{-\kappa-1}$$

with  $\beta = 1$  and  $2$ , respectively.

**Proposition 8** For any  $\alpha, \beta$  and  $|t| < \frac{1}{\sqrt{2}} \min \left\{ \left| s \pm \sqrt{s^2 - 1} \right| \right\}$

$$\frac{(1 - 2st + 2t^2)^{2-\beta}}{(1 - 4st + 8s^2t^2 - 8st^3 + 4t^4)^{\alpha+1}} = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} S(n, j; \alpha, \beta) t^n s^{n-2j}$$

where  $S(n, j; \alpha, \beta)$  is given in Definition 6.

**Proof.** Denote the left hand side by  $G(s, t; \alpha, \beta)$ . The expansion process begins with

$$\begin{aligned} & (1 - 2st + 2t^2)^{-2\alpha-\beta} \left( 1 - \frac{4(1-s^2)t^2}{(1-2st+2t^2)^2} \right)^{-\alpha-1} \\ &= \sum_{\ell=0}^{\infty} \frac{(\alpha+1)_{\ell}}{\ell!} 2^{2\ell} (1-s^2)^{\ell} t^{2\ell} (1-2st+2t^2)^{-2\alpha-\beta-2\ell}. \end{aligned}$$

By a variant of the generating function for Gegenbauer polynomials with  $\lambda > 0$

$$\begin{aligned} (1 - 2st + 2t^2)^{-\lambda} &= (1 + 2t^2)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} (2st)^k (1 + 2t^2)^{-k} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\lambda)_k}{k!} (2st)^k \frac{(\lambda+k)_m}{m!} (-2t^2)^m \\ &= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(\lambda)_{n-m}}{(n-2m)!m!} (-1)^m 2^{n-m} s^{n-2m}, \end{aligned}$$

changing the summation index  $k = n - 2m$ . Combining the expressions

results in

$$\begin{aligned}
G(s, t; \alpha, \beta) &= \sum_{\ell=0}^{\infty} \frac{(\alpha+1)_{\ell}}{\ell!} (2t)^{2\ell} (1-s^2)^{\ell} \sum_{k=0}^{\infty} t^k \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(2\alpha+\beta+2\ell)_{k-m}}{(k-2m)!m!} (-1)^m 2^{k-m} s^{k-2m} \\
&= \sum_{n=0}^{\infty} t^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{(\alpha+1)_{\ell}}{\ell!} (1-s^2)^{\ell} \sum_{m=0}^{\lfloor n/2 \rfloor - \ell} \frac{(2\alpha+\beta+2\ell)_{n-2\ell-m}}{(n-2\ell-2m)!m!} (-1)^m 2^{n-m} s^{n-2m-2\ell},
\end{aligned}$$

changing the summation indices to  $k = n - 2\ell$ . Expand  $(1-s^2)^{\ell} = \sum_{i=0}^{\ell} \binom{\ell}{i} (-s^2)^{\ell-i}$  and change indices replacing  $m$  by  $j-i$ . Then

$$\begin{aligned}
G(s, t; \alpha, \beta) &= \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\lfloor n/2 \rfloor} s^{n-2j} \\
&\times \sum_{\ell=0}^{\lfloor n/2 \rfloor} \sum_{i=\max(0, \ell+j-\lfloor n/2 \rfloor)}^{\min(\ell, j)} \frac{(\alpha+1)_{\ell} (2\alpha+\beta+2\ell)_{n-2\ell-j+i}}{i! (\ell-i)! (j-i)! (n-2\ell-2j+2i)!} (-1)^{\ell+j} 2^{n-j+i}.
\end{aligned}$$

The summation limits on  $i$  are derived from the bounds  $0 \leq i \leq \ell$ ,  $0 \leq i \leq j$ , and  $n - 2\ell - 2j + 2i \geq 0$ . the last bound implies  $i \geq \ell + j - \frac{n}{2}$  (if  $n = 2m + 1$  the bound is  $i \geq \ell + j - m$  and  $m = \lfloor n/2 \rfloor$ ). The bounds for  $s, t$  imply that the two factors  $(1 - 2xst + x^2t^2)$  for  $x = 1 \pm i$  do not vanish for  $|\sqrt{2}t| < \min \left| s \pm \sqrt{s^2 - 1} \right|$  and this is sufficient for convergence of the series (if  $s = \frac{1}{2}(z + z^{-1})$  then the convergence requirement is  $|\sqrt{2}t| < \min(|z|, |z|^{-1})$ ). This completes the proof for the formula for  $S(n, j; \alpha, \beta)$ . ■

Investigating harmonic polynomials in  $p_1, p_2, p_3$  which are  $\langle \cdot, \cdot \rangle_{\kappa}$ -orthogonal to the planar polynomials might be a plausible topic for further research.

## References

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